MA20218: ANALYSIS 2A

Chapter 0: Review from MA10207 and some basic results

0.1. Sequences.

Definition 0.1 (Convergence of sequences). *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $(\alpha_n)_{n\in\mathbb{N}}$ converges to a limit $L \in \mathbb{R}$, denoted $L = \lim_{n\to\infty} \alpha_n$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|\alpha_n - L| < \varepsilon \quad \forall\, n \geq N.$$

Equivalently, if

$$L - \varepsilon < \alpha_n < L + \varepsilon \quad \forall\, n \geq N. \tag{0.1}$$

We say that $(\alpha_n)_{n\in\mathbb{N}}$ diverges to $+\infty$ if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$\alpha_n > M \quad \forall\, n \geq N.$$

Similarly, we say that $(\alpha_n)_{n\in\mathbb{N}}$ diverges to $-\infty$ if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$\alpha_n < M \quad \forall\, n \geq N.$$

Proposition 0.2 (Cauchy criterion for convergence). *The sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges if and only if for each $\epsilon > 0$, there exists N such that*

$$|\alpha_m - \alpha_n| < \epsilon \;\; \forall m, n \geq N.$$

Remark 0.3. *Recall that, for a non-empty set $A \subset \mathbb{R}$, the infimum of A is denoted by* $\inf A$ *and is the greatest lower bound for A. The infimum (which need not lie in A) is an element of $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ which is less than or equal to any element in A and arbitrarily close to elements of A. Similarly,* $\sup A$ *(which need not lie in A) is the least upper bound for A and is an element of $\overline{\mathbb{R}}$ which is greater than or equal to any element in A and arbitrarily close to elements of A.*

Although not every real sequence converges, we can always define the largest and smallest accumulation points of a sequence:

Definition 0.4 (Limit superior and limit inferior). *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a real sequence.*

We define the limit superior of the sequence by

$$\limsup_{n\to\infty} \alpha_n = \lim_{n\to\infty} \left(\sup\{\alpha_m : m \geq n\}\right) = \inf_{n\in\mathbb{N}} \left(\sup\{\alpha_m : m \geq n\}\right),$$

We define the limit inferior of the sequence by

$$\liminf_{n\to\infty} \alpha_n = \lim_{n\to\infty} \left(\inf\{\alpha_m : m \geq n\}\right) = \sup_{n\in\mathbb{N}} \left(\inf\{\alpha_m : m \geq n\}\right),$$

Note that

$$\liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n.$$

If $\liminf_{n\to\infty} \alpha_n = \limsup_{n\to\infty} \alpha_n = L$, then the sequence converges and we have that $L = \lim_{n\to\infty} \alpha_n$.

Proposition 0.5 (Properties of lim sup and lim inf). *Let $(\alpha_n)_n$ be a real sequence; let $L_1 = \liminf_{n\to\infty} \alpha_n$ and $L_2 := \limsup_{n\to\infty} \alpha_n$. Then*

1. *For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

   $$\alpha_n > L_1 - \varepsilon \quad \forall\, n \geq N,$$

   namely only a finite number of elements is smaller than $L_1 - \varepsilon$.
2. *For every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that*

   $$\alpha_n < L_1 + \varepsilon.$$

Similarly

(3) *For every* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$\alpha_n < L_2 + \varepsilon \quad \forall\, n \geq N,$$

namely only a finite number of elements is larger than $L_2 + \varepsilon$*.*

(4) *For every* $\varepsilon > 0$ *there exist infinitely many* $n \in \mathbb{N}$ *such that*

$$\alpha_n > L_2 - \varepsilon.$$

Remark 0.6. *It can be shown using the last proposition that*

$$\limsup_{n\to\infty} \alpha_n := \sup\left(\lim_{k\to\infty} \alpha_{n_k} : (\alpha_{n_k})_k \text{ is a convergent subsequence}\right)$$

so that $\limsup_{n\to\infty} \alpha_n$ *is the supremum of the subsequential limits of* (α_n) *or, equivalently, the largest cluster point of the sequence. Similarly,*

$$\liminf_{n\to\infty} \alpha_n := \inf\left(\lim_{k\to\infty} \alpha_{n_k} : (\alpha_{n_k})_k \text{ is a convergent subsequence}\right)$$

so that $\liminf_{n\to\infty} \alpha_n$ *is the infimum of the subsequential limits of* (α_n) *or, equivalently, the smallest cluster point of the sequence.*

To compute limits of real sequences, it can be helpful to remember some basic results:

Proposition 0.7. *The following limits hold:*

- *Given* $a \in \mathbb{R}$*,*

$$\lim_{n\to+\infty} a^n = \begin{cases} +\infty & \text{if } a > 1, \\ 1 & \text{if } a = 1, \\ 0 & \text{if } -1 < a < 1, \end{cases}$$

  and it does not exist for $a \leq -1$*.*
- *For every* $a > 0$ *we have*

$$\lim_{n\to+\infty} a^{\frac{1}{n}} = 1;$$

- *For* $b \in \mathbb{R}$ *we have*

$$\lim_{n\to+\infty} n^b = \begin{cases} +\infty & \text{if } b > 0, \\ 1 & \text{if } b = 0, \\ 0 & \text{if } b < 0. \end{cases}$$

- *For every* $b \in \mathbb{R}$*:*

$$\lim_{n\to+\infty} (n^b)^{\frac{1}{n}} = 1;$$

- *For every* $b > 0$ *and* $a > 1$ *we have*

$$\lim_{n\to+\infty} \ln n = \lim_{n\to+\infty} n^b = \lim_{n\to+\infty} a^n = \lim_{n\to+\infty} n! = \lim_{n\to+\infty} n^n = +\infty;$$

- *For every* $b > 0$ *and* $a > 1$ *we have*

$$\lim_{n\to+\infty} \frac{\ln n}{n^b} = \lim_{n\to+\infty} \frac{n^b}{a^n} = \lim_{n\to+\infty} \frac{a^n}{n!} = \lim_{n\to+\infty} \frac{n!}{n^n} = 0.$$

0.2. **Series.**

We recall the main convergence criteria and some tests relating to convergence of series in $\mathbb{R}$.

Definition 0.8 (Convergence of series). *Let (α_k) be a sequence of real numbers, then the series*

$$\sum_{k=1}^{\infty} \alpha_k$$

is said to converge to the sum $s \in \mathbb{R}$ if and only if the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ defined by

$$s_n = \sum_{k=1}^{n} \alpha_k$$

converges to s as $n \to \infty$.
The series $\sum_{k=1}^{\infty} \alpha_k$ is said to be absolutely convergent if $\sum_{k=1}^{\infty} |\alpha_k|$ is a convergent series. A series which is convergent but not absolutely convergent is said to be conditionally convergent.

Proposition 0.9 (Cauchy criterion for convergence of a series.). *The series $\sum_{k=1}^{\infty} \alpha_k$ converges if and only if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\left| \sum_{k=N}^{N+p} \alpha_k \right| < \epsilon \quad \forall p \in \mathbb{N}.$$

Corollary 0.10. *If the series $\sum_{k=1}^{\infty} \alpha_k$ is absolutely convergent, then it is convergent.*

Theorem 0.11 (Vanishing test). *Let (α_k) be a sequence of real numbers. If the series*

$$\sum_{k=1}^{\infty} \alpha_k$$

converges (to a finite limit), then

$$\lim_{k\to\infty} \alpha_k = 0.$$

Theorem 0.12 (Comparison test). *Suppose that $\sum_{k=1}^{\infty} \alpha_k$ and $\sum_{k=1}^{\infty} \beta_k$ are two series satisfying*

$$0 \le \alpha_k \le \beta_k \ \forall k \in \mathbb{N}.$$

If $\sum_{k=1}^{\infty} \beta_k$ converges , then $\sum_{k=1}^{\infty} \alpha_k$ converges.

Theorem 0.13 (Ratio test). *Let (α_k) be a sequence of real numbers such that α_k is nonzero for k sufficiently large and let*

$$L_1 = \liminf_{k\to\infty} \left| \frac{\alpha_{k+1}}{\alpha_k} \right|, \quad \textit{and} \quad L_2 = \limsup_{k\to\infty} \left| \frac{\alpha_{k+1}}{\alpha_k} \right|.$$

Then the series

$$\sum_{k=1}^{\infty} \alpha_k$$

converges absolutely (to a finite limit) if $0 \le L_2 < 1$ and diverges or does not converge if $1 < L_1 \le +\infty$.

We now state another convergence criterion, the ratio test, which is formulated in terms of the limsup.

Theorem 0.14 (Root test). *Let (α_k) be a sequence of real numbers, and let*

$$\gamma = \limsup_{k\to\infty} |\alpha_k|^{\frac{1}{k}}.$$

Then the series

$$\sum_{k=1}^{\infty} \alpha_k$$

converges absolutely (to a finite limit) if $0 \le \gamma < 1$ and diverges or does not converge if $1 < \gamma \le +\infty$.

Theorem 0.15 (The integral test)**.** *Suppose that $f : [1, \infty) \to \mathbb{R}$ is a positive decreasing continuous function satisfying $\lim_{x\to\infty} f(x) = 0$. Then the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the sequence $(\beta_n)_{n\in\mathbb{N}}$ of integrals*

$$\beta_n = \int_1^n f(x)\,dx \quad n \in \mathbb{N}$$

converges as $n \to \infty$.

Theorem 0.16 (Leibniz or alternating series test)**.** *Suppose that (α_n) is a monotonically decreasing sequence of non-negative terms converging to zero. (i.e., $\alpha_1 \geq \alpha_2 ... \geq \alpha_n \geq \alpha_{(n+1)} \geq ... \geq 0$ and $\alpha_k \to 0$ as $k \to \infty$.) Then the series*

$$\sum_{k=1}^{\infty} (-1)^k \alpha_k$$

converges.

Remark 0.17. *The last result provides a method of generating conditionally convergent series.*

0.3. **Continuity and integrability.**

Definition 0.18 (Continuous function)**.** *Let $A \subset \mathbb{R}$ be a set and $f : A \to \mathbb{R}$ a function. Suppose that $x \in A$. We say that f is continuous at x if*

$$\forall \varepsilon\ \exists \delta > 0 \quad \text{such that } \forall y \in A : |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon.$$

Equivalently, in terms of sequences, we have that f is continuous at x if for any sequence $(x_k)_{k\in\mathbb{N}}$ in A with $x = \lim_{k\to\infty} x_k$, we have

$$f(x) = \lim_{k\to\infty} f(x_k).$$

We say that f is continuous on A if it is continuous at every point of A.

We say that f is uniformly continuous on A if

$$\forall \varepsilon\ \exists \delta > 0 \quad \text{such that } \forall x, y \in A : |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon.$$

Finally, we say that f is Lipschitz continuous if there exists a number $L > 0$ such that for all $x, y \in A$,

$$|f(y) - f(x)| \leq L|y - x|.$$

These definitions may have been given in MA10207 only for functions defined on an interval. The conditions are exactly the same, however, for any set $A \subset \mathbb{R}$. If A is a closed, bounded interval, then continuity has particularly nice consequences.

Theorem 0.19 (Theorem of uniform continuity)**.** *Let $I \subset \mathbb{R}$ be a closed, bounded interval and suppose that $f : I \to \mathbb{R}$ is continuous. Then f is uniformly continuous on I.*

Theorem 0.20 (Weierstrass extreme value theorem)**.** *Let $I \subset \mathbb{R}$ be a closed, bounded interval and suppose that $f : I \to \mathbb{R}$ is continuous. Then f is bounded and attains its infimum and supremum.*

The notation we will use for Riemann integration (which may differ slightly from what you have seen last year) is the following.

Definition 0.21. *A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable on $[a, b]$ if for every $\varepsilon > 0$ there exists a subdivision Δ of $[a, b]$, $\Delta = \{a = x_0 < x_1 < \cdots < x_M = b\}$ such that*

$$U(f, \Delta) - L(f, \Delta) = \sum_{n=1}^{M} \left(\sup_{I_n} f - \inf_{I_n} f \right) |I_n| < \varepsilon, \quad I_n = x_{n+1} - x_n. \tag{0.2}$$

We also denote with $\omega(f, I_n) := \sup_{I_n} f - \inf_{I_n} f$ the oscillation of f on I_n; in this case (0.2) can be rewritten as

$$U(f, \Delta) - L(f, \Delta) = \sum_{n=1}^{M} \omega(f, I_n)\,|I_n| < \varepsilon.$$